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Some approximation properties of (p,q)-Bernstein operators

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Abstract
This paper is concerned with the (p,q)-analog of Bernstein operators. It is proved that, when the function is convex, the (p,q)-Bernstein operators are monotonic decreasing, as in the classical case. Also, some numerical examples based on Maple algorithms that verify these properties are considered. A global approximation theorem by means of the Ditzian-Totik modulus of smoothness and a Voronovskaja type theorem are proved.

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1 Introduction and preliminaries

During the last decade, the applications of q-calculus in the field of approximation theory has led to the discovery of new generalizations of classical operators. Lupaş [1] was first to observe the possibility of using q-calculus in this context. For more comprehensive details the reader should consult monograph of Aral *et al.* [2] and the recent references [3–9].

Nowadays, the generalizations of several operators in post-quantum calculus, namely the (p,q)-calculus have been studied intensively. The (p,q)-calculus has been used in many areas of sciences, such as oscillator algebra, Lie group theory, field theory, differential equations, hypergeometric series, physical sciences (see [10, 11]). Recently, Mursaleen *et al.* [12] defined (p,q)-analog of Bernstein operators. The approximation properties for these operators based on Korovkin's theorem and some direct theorems were considered. Also, many well-known approximation operators have been introduced using these techniques, such as Bleimann-Butzer-Hahn operators [13] and Szász-Mirakyan operators [14].

In the present paper, we prove new approximation properties of (p,q)-analog of Bernstein operators. First of all, we recall some notations and definitions from the (p,q)-calculus. Let $0 < q < p \le 1$. For each non-negative integer $n \ge k \ge 0$, the (p,q)-integer $[k]_{p,q}$, (p,q)-factorial $[k]_{p,q}!$, and (p,q)-binomial are defined by

$$[k]_{p,q} := \frac{p^k - q^k}{p-q},$$

$$[k]_{p,q}! := \begin{cases} [k]_{p,q}[k-1]_{p,q}\cdots[1]_{p,q}, & k \ge 1, \\ 1, & k = 0, \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} := \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}.$$

As a special case when $p = 1$, the above notations reduce to q-analogs.

The (p,q)-power basis is defined as

$$(x \ominus a)^n_{p,q} = (x-a)(px-qa)\left(p^2x-q^2a\right)\cdots\left(p^{n-1}x-q^{n-1}a\right).$$

The (p,q)-derivative of the function f is defined as

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, \quad x \neq 0.$$

Let f be an arbitrary function and $a \in \mathbb{R}$. The (p,q)-integral of f on $[0,a]$ is defined as

$$\int_0^a f(t)d_{p,q}t = (q-p)a\sum_{k=0}^{\infty} f\left(\frac{p^k}{q^{k+1}}a\right)\frac{p^k}{q^{k+1}}, \quad \text{if } \left|\frac{p}{q}\right| < 1,$$

$$\int_0^a f(t)d_{p,q}t = (p-q)a\sum_{k=0}^{\infty} f\left(\frac{q^k}{p^{k+1}}a\right)\frac{q^k}{p^{k+1}}, \quad \text{if } \left|\frac{q}{p}\right| < 1.$$

The (p,q)-analog of Bernstein operators for $x \in [0,1]$ and $0 < q < p \leq 1$ are introduced as follows:

$$B_n^{p,q}(f;x) = \sum_{k=0}^{n} b_{n,k}^{p,q}(x) f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right),$$

where the (p,q)-Bernstein basis is defined as

$$b_{n,k}^{p,q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{[k(k-1)-n(n-1)]/2} x^k (1 \ominus x)^{n-k}_{p,q}.$$

Lemma 1.1 *For $x \in [0,1]$, $0 < q < p \leq 1$, we have*

$$B_n^{p,q}(e_0;x) = 1, \qquad B_n^{p,q}(e_1;x) = x,$$

$$B_n^{p,q}(e_2;x) = \frac{p^{n-1}}{[n]_{p,q}}x + \frac{q[n-1]_{p,q}}{[n]_{p,q}}x^2,$$

where $e_i(x) = x^i$ and $i \in \{0,1,2\}$.

Lemma 1.2 *Let n be a given natural number, then*

$$B_n^{p,q}\left((t-x)^2;x\right) = \frac{p^{n-1}}{[n]_{p,q}}\phi^2(x) \leq \frac{1}{[n]_{p,q}}\phi^2(x),$$

where $\phi(x) = \sqrt{x(1-x)}$ and $x \in [0,1]$.

2 Monotonicity for convex functions

Oru and Phillips [15] proved that when the function f is convex on $[0,1]$, its q-Bernstein operators are monotonic decreasing. In this section we will study the monotonicity of (p,q)-Bernstein operators.

Theorem 2.1 *If f is convex function on* $[0,1]$, *then*

$$B_n^{p,q}(f;x) \geq f(x), \quad 0 \leq x \leq 1,$$

for all $n \geq 1$ and $0 < q < p \leq 1$.

Proof We consider the knots $x_k = \frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}$, $\lambda_k = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_{p,q} p^{[k(k-1)-n(n-1)]/2} x^k (1 \ominus x)_{p,q}^{n-k}$, $0 \leq k \leq n$. Using Lemma 1.1, it follows that

$$\lambda_0 + \lambda_1 + \cdots + \lambda_n = 1,$$

$$x_0\lambda_0 + x_1 + \lambda_1 + \cdots + x_n\lambda_n = x.$$

From the convexity of the function f, we get

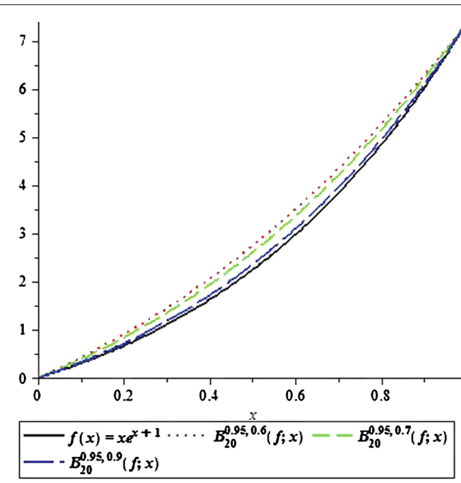
$$B_n^{p,q}(f;x) = \sum_{k=0}^{n} \lambda_k f(x_k) \geq f\left(\sum_{k=0}^{n} \lambda_k x_k\right) = f(x).$$

□

Example 2.2 Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = xe^{x+1}$. Figure 1 illustrates that $B_n^{p,q}(f;x) \geq f(x)$ for the convex function f and $x \in [0,1]$.

Theorem 2.3 *Let f be convex on* $[0,1]$. *Then* $B_{n-1}^{p,q}(f;x) \geq B_n^{p,q}(f;x)$ *for* $0 < q < p \leq 1$, $0 \leq x \leq 1$, *and* $n \geq 2$. *If* $f \in C[0,1]$ *the inequality holds strictly for* $0 < x < 1$ *unless f is linear in each of the intervals between consecutive knots* $\frac{p^{n-1-k}[k]_{p,q}}{[n-1]_{p,q}}$, $0 \leq k \leq n-1$, *in which case we have the equality.*

Figure 1 **Approximation process by $B_n^{p,q}(f;x)$ for $f(x) = xe^{x+1}$.**



Proof For $0 < q < p \leq 1$ we begin by writing

$$\begin{aligned}
&\prod_{s=0}^{n-1}\left(p^s - q^s x\right)^{-1}\left[B_{n-1}^{p,q}(f;x) - B_n^{p,q}(f;x)\right] \\
&\quad = \prod_{s=0}^{n-1}\left(p^s - q^s x\right)^{-1}\left[\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{[k(k-1)-(n-2)(n-1)]/2} x^k (1\ominus x)_{p,q}^{n-k-1} f\left(\frac{p^{n-1-k}[k]}{[n-1]}\right)\right. \\
&\qquad \left. - \sum_{k=0}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{[k(k-1)-n(n-1)]/2} x^k (1\ominus x)_{p,q}^{n-k} f\left(\frac{p^{n-k}[k]}{[n]}\right)\right] \\
&\quad = \sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{[k(k-1)-(n-2)(n-1)]/2} x^k \prod_{s=n-k-1}^{n-1}\left(p^s - q^s x\right)^{-1} f\left(\frac{p^{n-1-k}[k]}{[n-1]}\right) \\
&\qquad - \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{[k(k-1)-n(n-1)]/2} x^k \prod_{s=n-k}^{n-1}\left(p^s - q^s x\right)^{-1} f\left(\frac{p^{n-k}[k]}{[n]}\right).
\end{aligned}$$

Denote

$$\Psi_k(x) = p^{\frac{k(k-1)}{2}} x^k \prod_{s=n-k}^{n-1}\left(p^s - q^s x\right)^{-1}, \tag{2.1}$$

and using the following relation:

$$p^{n-1} p^{\frac{k(k-1)}{2}} x^k \prod_{s=n-k-1}^{n-1}\left(p^s - q^s x\right)^{-1} = p^k \Psi_k(x) + q^{n-k-1}\Psi_{k+1}(x),$$

we find

$$\begin{aligned}
&\prod_{s=0}^{n-1}\left(p^s - q^s x\right)^{-1}\left[B_{n-1}^{p,q}(f;x) - B_n^{p,q}(f;x)\right] \\
&\quad = \sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^{-\frac{(n-1)(n-2)}{2}} p^{-(n-1)}\left\{p^k\Psi_k(x) + q^{n-k-1}\Psi_{k+1}(x)\right\} f\left(\frac{p^{n-1-k}[k]_{p,q}}{[n-1]_{p,q}}\right) \\
&\qquad - \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{-\frac{n(n-1)}{2}}\Psi_k(x) f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right) \\
&\quad = p^{-\frac{n(n-1)}{2}}\left\{\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^k\Psi_k(x) f\left(\frac{p^{n-1-k}[k]_{p,q}}{[n-1]_{p,q}}\right)\right. \\
&\qquad \left. + \sum_{k=1}^{n}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} q^{n-k}\Psi_k(x) f\left(\frac{p^{n-k}[k-1]_{p,q}}{[n-1]_{p,q}}\right) - \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\Psi_k(x) f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)\right\} \\
&\quad = p^{-\frac{n(n-1)}{2}}\sum_{k=1}^{n-1}\left\{\begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} p^k f\left(\frac{p^{n-1-k}[k]_{p,q}}{[n-1]_{p,q}}\right) + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} q^{n-k} f\left(\frac{p^{n-k}[k-1]_{p,q}}{[n-1]_{p,q}}\right)\right. \\
&\qquad \left. - \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right)\right\}\Psi_k(x)
\end{aligned}$$

$$= p^{-\frac{n(n-1)}{2}} \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left\{ \frac{[n-k]_{p,q}}{[n]_{p,q}} p^k f\left(\frac{p^{n-1-k}[k]_{p,q}}{[n-1]_{p,q}} \right) + \frac{[k]_{p,q}}{[n]_{p,q}} q^{n-k} f\left(\frac{p^{n-k}[k-1]_{p,q}}{[n-1]_{p,q}} \right) - f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}} \right) \right\} \Psi_k(x)$$

$$= p^{-\frac{n(n-1)}{2}} \sum_{k=1}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a_k \Psi_k(x),$$

where

$$a_k = \frac{[n-k]_{p,q}}{[n]_{p,q}} p^k f\left(\frac{p^{n-1-k}[k]_{p,q}}{[n-1]_{p,q}} \right) + \frac{[k]_{p,q}}{[n]_{p,q}} q^{n-k} f\left(\frac{p^{n-k}[k-1]_{p,q}}{[n-1]_{p,q}} \right) - f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}} \right).$$

From (2.1) it is clear that each $\Psi_k(x)$ is non-negative on $[0,1]$ for $0 < q < p \leq 1$ and, thus, it suffices to show that each a_k is non-negative.

Since f is convex on $[0,1]$, then for any $t_0, t_1 \in [0,1]$ and $\lambda \in [0,1]$, it follows that

$$f\bigl(\lambda t_0 + (1-\lambda)t_1\bigr) \leq \lambda f(t_0) + (1-\lambda) f(t_1).$$

If we choose $t_0 = \frac{p^{n-k}[k-1]_{p,q}}{[n-1]_{p,q}}$, $t_1 = \frac{p^{n-1-k}[k]_{p,q}}{[n-1]_{p,q}}$, and $\lambda = \frac{[k]_{p,q}}{[n]_{p,q}} q^{n-k}$, then $t_0, t_1 \in [0,1]$ and $\lambda \in (0,1)$ for $1 \leq k \leq n-1$, and we deduce that

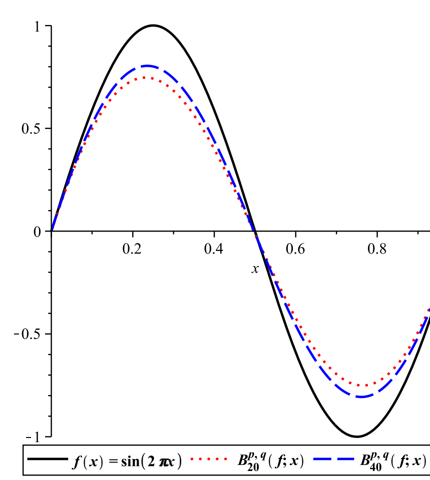
$$a_k = \lambda f(t_0) + (1-\lambda) f(t_1) - f\bigl(\lambda t_0 + (1-\lambda)t_1\bigr) \geq 0.$$

Thus $B_{n-1}^{p,q}(f;x) \geq B_n^{p,q}(f;x)$.

We have equality for $x = 0$ and $x = 1$, since the Bernstein polynomials interpolate f on these end-points. The inequality will be strict for $0 < x < 1$ unless when f is linear in each of the intervals between consecutive knots $\frac{p^{n-1-k}[k]_{p,q}}{[n-1]_{p,q}}$, $0 \leq k \leq n-1$, then we have $B_{n-1}^{p,q}(f;x) = B_n^{p,q}(f;x)$ for $0 \leq x \leq 1$. □

Example 2.4 Let $f(x) = \sin(2\pi x)$, $x \in [0,1]$. Figure 2 illustrates the monotonicity of (p,q)-Bernstein operators for $p = 0.95$ and $q = 0.9$. We note that if f is increasing (decreasing) on $[0,1]$, then the operators is also increasing (decreasing) on $[0,1]$.

Figure 2 Monotonicity of (p,q)-Bernstein operators.



3 A global approximation theorem

In the following we establish a global approximation theorem by means of Ditzian-Totik modulus of smoothness. In order to prove our next result, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional [16]. Let $\phi(x) = \sqrt{x(1-x)}$ and $f \in C[0,1]$. The first order modulus of smoothness is given by

$$\omega_\phi(f;t) = \sup_{0<h\leq t}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right)-f\left(x-\frac{h\phi(x)}{2}\right)\right|, x\pm\frac{h\phi(x)}{2}\in[0,1]\right\}. \tag{3.1}$$

The corresponding K-functional to (3.1) is defined by

$$K_\phi(f;t) = \inf_{g\in W_\phi[0,1]}\left\{\|f-g\| + t\left\|\phi g'\right\|\right\} \quad (t>0),$$

where $W_\phi[0,1] = \{g : g \in AC_{\text{loc}}[0,1], \|\phi g'\| < \infty\}$ and $g \in AC_{\text{loc}}[0,1]$ means that g is absolutely continuous on every interval $[a,b] \subset [0,1]$. It is well known ([16], p.11) that there exists a constant $C > 0$ such that

$$K_\phi(f;t) \leq C\omega_\phi(f;t). \tag{3.2}$$

Theorem 3.1 *Let $f \in C[0,1]$ and $\phi(x) = \sqrt{x(1-x)}$, then for every $x \in [0,1]$, we have*

$$\left|B_n^{p,q}(f;x) - f(x)\right| \leq C\omega_\phi\left(f;\frac{1}{\sqrt{[n]_{p,q}}}\right),$$

where C is a constant independent of n and x.

Proof Using the representation

$$g(t) = g(x) + \int_x^t g'(u)\,du,$$

we get

$$\left|B_n^{p,q}(g;x) - g(x)\right| = \left|B_n^{p,q}\left(\int_x^t g'(u)\,du;x\right)\right|. \tag{3.3}$$

For any $x \in (0,1)$ and $t \in [0,1]$ we find that

$$\left|\int_x^t g'(u)\,du\right| \leq \left\|\phi g'\right\|\left|\int_x^t \frac{1}{\phi(u)}\,du\right|. \tag{3.4}$$

Further,

$$\begin{aligned}\left|\int_x^t \frac{1}{\phi(u)}\,du\right| &= \left|\int_x^t \frac{1}{\sqrt{u(1-u)}}\,du\right| \\ &\leq \left|\int_x^t \left(\frac{1}{\sqrt{u}} + \frac{1}{\sqrt{1-u}}\right)du\right| \\ &\leq 2\left(|\sqrt{t}-\sqrt{x}| + |\sqrt{1-t}-\sqrt{1-x}|\right)\end{aligned}$$

$$= 2|t-x|\left(\frac{1}{\sqrt{t}+\sqrt{x}}+\frac{1}{\sqrt{1-t}+\sqrt{1-x}}\right)$$
$$< 2|t-x|\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\right) \leq \frac{2\sqrt{2}|t-x|}{\phi(x)}. \tag{3.5}$$

From (3.3)-(3.5) and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|B_n^{p,q}(g;x)-g(x)\right| &< 2\sqrt{2}\left\|\phi g'\right\|\phi^{-1}(x)B_n^{p,q}\left(|t-x|;x\right)\\ &\leq 2\sqrt{2}\left\|\phi g'\right\|\phi^{-1}(x)\left(B_n^{p,q}\left((t-x)^2;x\right)\right)^{1/2}.\end{aligned}$$

Using Lemma 1.2, we get

$$\left|B_n^{p,q}(g;x)-g(x)\right| \leq \frac{2\sqrt{2}}{\sqrt{[n]_{p,q}}}\left\|\phi g'\right\|.$$

Now, using the above inequality we can write

$$\begin{aligned}\left|B_n^{p,q}(f;x)-f(x)\right| &\leq \left|B_n^{p,q}(f-g;x)\right| + \left|f(x)-g(x)\right| + \left|B_n^{p,q}(g;x)-g(x)\right|\\ &\leq 2\sqrt{2}\left(\|f-g\| + \frac{1}{\sqrt{[n]_{p,q}}}\left\|\phi g'\right\|\right).\end{aligned}$$

Taking the infimum on the right-hand side of the above inequality over all $g \in W_\phi[0,1]$, we get

$$\left|B_n^{p,q}(f;x)-f(x)\right| \leq CK_\phi\left(f;\frac{1}{\sqrt{[n]_{p,q}}}\right).$$

Using equation (3.2) this theorem is proven. □

4 Voronovskaja type theorem

Using the first order Ditzian-Totik modulus of smoothness, we prove a quantitative Voronovskaja type theorem for the (p,q)-Bernstein operators.

Theorem 4.1 *For any $f \in C^2[0,1]$ the following inequalities hold*:
(i) $|[n]_{p,q}[B_n^{p,q}(f;x)-f(x)] - \frac{p^{n-1}\phi^2(x)}{2}f''(x)| \leq C\omega_\phi(f'',\phi(x)n^{-1/2})$,
(ii) $|[n]_{p,q}[B_n^{p,q}(f;x)-f(x)] - \frac{p^{n-1}\phi^2(x)}{2}f''(x)| \leq C\phi(x)\omega_\phi(f'',n^{-1/2})$,
where C is a positive constant.

Proof Let $f \in C^2[0,1]$ be given and $t,x \in [0,1]$. Using Taylor's expansion, we have

$$f(t)-f(x) = (t-x)f'(x) + \int_x^t (t-u)f''(u)\,du.$$

Therefore,

$$\begin{aligned}f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}(t-x)^2f''(x) &= \int_x^t (t-u)f''(u)\,du - \int_x^t (t-u)f''(x)\,du\\ &= \int_x^t (t-u)\left[f''(u)-f''(x)\right]du.\end{aligned}$$

In view of Lemma 1.1 and Lemma 1.2, we get

$$\left|B_n^{p,q}(f;x)-f(x)-\frac{p^{n-1}}{2[n]_{p,q}}\phi^2(x)f''(x)\right|\le B_n^{p,q}\left(\left|\int_x^t|t-u|\left|f''(u)-f''(x)\right|du\right|;x\right). \tag{4.1}$$

The quantity $|\int_x^t|f''(u)-f''(x)||t-u|\,du|$ was estimated in [17], p.337, as follows:

$$\left|\int_x^t\left|f''(u)-f''(x)\right||t-u|\,du\right|\le 2\left\|f''-g\right\|(t-x)^2+2\left\|\phi g'\right\|\phi^{-1}(x)|t-x|^3, \tag{4.2}$$

where $g\in W_\phi[0,1]$. On the other hand, for any $m=1,2,\dots$ and $0<q<p\le 1$, there exists a constant $C_m>0$ such that

$$\left|B_n^{p,q}\left((t-x)_{p,q}^m;x\right)\right|\le C_m\frac{\phi^2(x)}{[n]_{p,q}^{\lfloor\frac{m+1}{2}\rfloor}}, \tag{4.3}$$

where $x\in[0,1]$ and $\lfloor a\rfloor$ is the integer part of $a\ge 0$.

Throughout this proof, C denotes a constant not necessarily the same at each occurrence.

Now, combining (4.1)-(4.3) and applying Lemma 1.2, the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\left|B_n^{p,q}(f;x)-f(x)-\frac{p^{n-1}\phi^2(x)}{2[n]_{p,q}}f''(x)\right|\\
&\quad\le 2\left\|f''-g\right\|B_n^{p,q}\left((t-x)^2;x\right)+2\left\|\phi g'\right\|\phi^{-1}(x)B_n^{p,q}\left(|t-x|^3;x\right)\\
&\quad\le 2\left\|f''-g\right\|\frac{\phi^2(x)}{[n]_{p,q}}+2\left\|\phi g'\right\|\phi^{-1}(x)\left\{B_n^{p,q}(t-x)^2;x\right\}^{1/2}\left\{B_n^{p,q}\left((t-x)^4;x\right)\right\}^{1/2}\\
&\quad\le 2\left\|f''-g\right\|\frac{\phi^2(x)}{[n]_{p,q}}+2\frac{C}{[n]_{p,q}}\left\|\phi g'\right\|\frac{\phi(x)}{\sqrt{[n]_{p,q}}}\\
&\quad\le\frac{C}{[n]_{p,q}}\left\{\phi^2(x)\left\|f''-g\right\|+[n]_{p,q}^{-1/2}\phi(x)\left\|\phi g'\right\|\right\}.
\end{aligned}$$

Since $\phi^2(x)\le\phi(x)\le 1$, $x\in[0,1]$, we obtain

$$\left|[n]_{p,q}\left[B_n^{p,q}(f;x)-f(x)\right]-\frac{p^{n-1}\phi^2(x)}{2}f''(x)\right|\le C\left\{\left\|f''-g\right\|+[n]_{p,q}^{-1/2}\phi(x)\left\|\phi g'\right\|\right\}.$$

Also, the following inequality can be obtained:

$$\left|[n]_{p,q}\left[B_n^{p,q}(f;x)-f(x)\right]-\frac{p^{n-1}\phi^2(x)}{2}f''(x)\right|\le C\phi(x)\left\{\left\|f''-g\right\|+[n]_{p,q}^{-1/2}\left\|\phi g'\right\|\right\}.$$

Taking the infimum on the right-hand side of the above relations over $g\in W_\phi[0,1]$, we get

$$\left|[n]_{p,q}\left[B_n^{p,q}(f;x)-f(x)\right]-\frac{p^{n-1}\phi^2(x)}{2}f''(x)\right|\le\begin{cases}CK_\phi(f'';\phi(x)[n]_{p,q}^{-1/2}),\\ C\phi(x)K_\phi(f'';[n]_{p,q}^{-1/2}).\end{cases} \tag{4.4}$$

Using (4.4) and (3.2) the theorem is proved. □

5 Better approximation

In 2003, King [18] proposed a technique to obtain a better approximation for the well-known Bernstein operators as follows:

$$\big((B_n f) \circ r_n\big)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}\big(r_n(x)\big)^k\big(1 - r_n(x)\big)^{n-k}, \tag{5.1}$$

where r_n is a sequence of continuous functions defined on $[0,1]$ with $0 \le r_n(x) \le 1$ for each $x \in [0,1]$ and $n \in \{1,2,\dots\}$. The modified Bernstein operators (5.1) preserve e_0 and e_2 and present a degree of approximation at least as good. In [19], the authors consider the sequence of linear Bernstein-type operators defined for $f \in C[0,1]$ by $B_n(f \circ \tau^{-1}) \circ \tau$, τ being any function that is continuously differentiable ∞ times on $[0,1]$, such that $\tau(0) = 0$, $\tau(1) = 1$, and $\tau'(x) > 0$ for $x \in [0,1]$.

So, using the technique proposed in [19], we modify the (p,q)-Bernstein operators as follows:

$$\overline{B}_n^{p,q}(f;x) = \sum_{k=0}^{n} \overline{b}_{n,k}^{p,q}(x)\big(f \circ \tau^{-1}\big)\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right),$$

where

$$\overline{b}_n^{p,q}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{[k(k-1)-n(n-1)]/2}\tau(x)^k\big(1 \ominus \tau(x)\big)_{p,q}^{n-k}.$$

Then we have

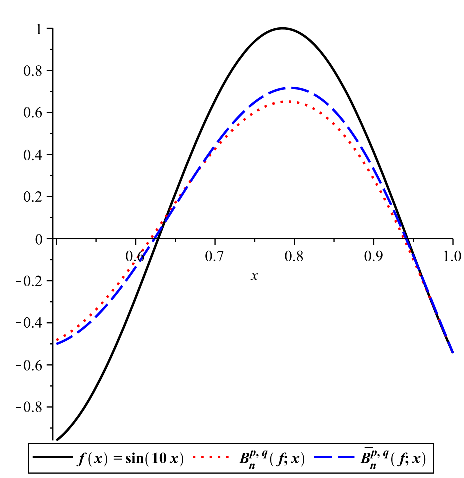
$$\overline{B}_n^{p,q}(e_0;x) = 1, \qquad \overline{B}_n^{p,q}\big(\tau(t);x\big) = \tau(x),$$

$$\overline{B}_n^{p,q}\big(\tau^2(t);x\big) = \frac{p^{n-1}}{[n]_{p,q}}\tau(x) + \frac{q[n-1]_{p,q}}{[n]_{p,q}}\tau^2(x),$$

$$\overline{B}_n^{p,q}\big(\big(\tau(t) - \tau(x)\big)^2;x\big) = \frac{p^{n-1}}{[n]_{p,q}}\phi_\tau^2(x),$$

where $\phi_\tau^2(x) := \tau(x)(1 - \tau(x))$.

Figure 3 Approximation process by $B_n^{p,q}$ and $\overline{B}_n^{p,q}$.



Example 5.1 We compare the convergence of (p,q)-analog of Bernstein operators $B_n^{p,q}f$ with the modified operators $\overline{B}_n^{p,q}f$. We have considered the function $f(x) = \sin(10x)$ and $\tau(x) = \frac{x^2+x}{2}$. For $x \in [\frac{1}{2}, 1]$, $p = 0.95$, $q = 0.9$, $n = 100$, the convergence of the operators $B_n^{p,q}$ and $\overline{B}_n^{p,q}$ to the function f is illustrated in Figure 3. Note that the approximation by $\overline{B}_n^{p,q}f$ is better than using (p,q)-Bernstein operators $B_n^{p,q}f$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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References
1. Lupaş, A: A q-analogue of the Bernstein operator. In: Seminar on Numerical and Statistical Calculus, vol. 9, pp. 85-92. University of Cluj-Napoca, Cluj-Napoca (1987)
2. Aral, A, Gupta, V, Agarwal, RP: Applications of q-Calculus in Operator Theory. Springer, New York (2013)
3. Acu, AM: Stancu-Schurer-Kantorovich operators based on q-integers. Appl. Math. Comput. **259**, 896-907 (2015). doi:10.1016/j.amc.2015.03.032
4. Acar, T, Aral, A: On pointwise convergence of q-Bernstein operators and their q-derivatives. Numer. Funct. Anal. Optim. **36**(3), 287-304 (2015). doi:10.1080/01630563.2014.970646
5. Acu, AM, Muraru, CV: Approximation properties of bivariate extension of q-Bernstein-Schurer-Kantorovich operators. Results Math. **67**(3-4), 265-279 (2015). doi:10.1007/s00025-015-0441-7
6. Agratini, O: On a q-analogue of Stancu operators. Cent. Eur. J. Math. **8**(1), 191-198 (2010). doi:10.2478/s11533-009-0057-9
7. Kang, SM, Acu, AM, Rafiq, A, Kwun, YC: Approximation properties of q-Kantorovich-Stancu operator. J. Inequal. Appl. **2015**, 211 (2015). doi:10.1186/s13660-015-0729-x
8. Kang, SM, Acu, AM, Rafiq, A, Kwun, YC: On q-analogue of Stancu-Schurer-Kantorovich operators based on q-Riemann integral. J. Comput. Anal. Appl. **21**(3), 564-577 (2016)
9. Ulusoy, G, Acar, T: q-Voronovskaya type theorems for q-Baskakov operators. Math. Methods Appl. Sci. (2015). doi:10.1002/mma.3784
10. Burban, I: Two-parameter deformation of the oscillator algebra and (p,q)-analog of two-dimensional conformal field theory. J. Nonlinear Math. Phys. **2**(3-4), 384-391 (1995). doi:10.2991/jnmp.1995.2.3-4.18
11. Sahai, V, Yadav, S: Representations of two parameter quantum algebras and p,q-special functions. J. Math. Anal. Appl. **335**, 268-279 (2007). doi:10.1016/j.jmaa.2007.01.072
12. Mursaleen, M, Ansari, KJ, Khan, A: Erratum to 'On (p,q)-analogue of Bernstein operators' [Appl. Math. Comput. 266 (2015) 874-882]. Appl. Math. Comput. **278**, 70-71 (2016). doi:10.1016/j/amc.2015.04.090
13. Mursaleen, M, Nasiruzzaman, M, Khan, A, Ansari, KJ: Some approximation results on Bleimann-Butzer-Hahn operators defined by (p,q)-integers. Filomat **30**(3), 639-648 (2016). doi:10.2298/FIL1603639M
14. Acar, T: (p,q)-Generalization of Szász-Mirakyan operators. Math. Methods Appl. Sci. (2015). doi:10.1002/mma.3721
15. Oru, H, Phillips, GM: A generalization of the Bernstein polynomials. Proc. Edinb. Math. Soc. **42**, 403-413 (1999). doi:10.1017/S0013091500020332
16. Ditzian, Z, Totik, V: Moduli of Smoothness. Springer, New York (1987)
17. Finta, Z: Remark on Voronovskaja theorem for q-Bernstein operators. Stud. Univ. Babeş-Bolyai, Math. **56**(2), 335-339 (2011)
18. King, JP: Positive linear operators which preserve x^2. Acta Math. Hung. **99**, 203-208 (2003). doi:10.1023/A:1024571126455
19. Cárdenas-Morales, D, Garrancho, P, Raşa, I: Bernstein-type operators which preserve polynomials. Comput. Math. Appl. **62**(1), 158-163 (2011). doi:10.1016/j.camwa.2011.04.063